The Riemannian Penrose inequality and a virtual gravitational collapse

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We reinterpret the proof of the Riemannian Penrose inequality by H. Bray. The modified argument turns out to have a nice feature so that the flow of Riemannian metrics appearing Bray's proof gives a Lorentzian metric of a spacetime. We also discuss a possible extension of our approach to charged black holes.

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I. INTRODUCTION

The issue of cosmic censorship is still an unsolved problem. Closely related to this, Penrose proposed the following inequality for the black hole [1]

$$\sqrt{A/16\pi} \leq m, \tag{1}$$

where A is the area of the horizon and m is the ADM mass for an asymptotically flat spacetime. This inequality is also yet to be proved and remains an important problem.

In a Riemannian/time-symmetric space, Huisken and Ilmanen proved this inequality where the area A is that of a single black hole by using the inverse mean curvature flow [2]. At almost the same time, Bray proved it for multi black holes using a conformal flow method [3]. For the general, non-time-symmetric case, the Penrose inequality is still an open question.

As we review in the next section, Bray's proof is a bit of a mystery. This is because it is difficult to have a physical reasoning why the proof works. In this paper, we introduce a normalised conformal flow and then we regard it as a model of the time evolution, formulating a Lorentzian metric. As a result, we have a rather natural interpretation of Bray's proof. We also discuss some implications of our line of reasoning to the charged black hole case.

The rest of this paper is organized as follows. In the next section, we review Bray's proof. In Sec. III, we present the modified proof. Then we give some physical interpretations of our new proof in Sec. IV. As an extension, we discuss the Penrose inequality for charged black holes in Sec. V. Finally we summarize our results in Sec. VI.

II. BRIEF SKETCH OF BRAY'S PROOF

We consider a time-symmetric initial data (Σ, q_0) where q_0 is a Riemannian metric. The time-symmetric initial data is defined by a hypersurface in a spacetime with the zero extrinsic curvature. We suppose that the apparent horizons H_0 exist in the spacetime. It is known that the apparent horizon corresponds to the minimal surface in (Σ, q_0).

We introduce the following conformal transformation

$$q_t = u_t^4 q_0 \tag{2}$$

and define v_t as the "time" derivative of u_t

$$v_t = \dot{u}_t, \tag{3}$$

where dot stands for the derivative with respect to the parameter t. We then require that v_t is a harmonic function with respect to q_0

$$\Delta_{q_0} v_t = 0 \tag{4}$$

with the boundary condition

$$v_t(x)|_{H_t} = 0 \tag{5}$$

and

$$v_t \to -e^{-t} \quad \text{as} \quad r \to \infty. \tag{6}$$

We require that H_t is the minimal surface in (Σ, q_t). From the definition of v_t, we have

$$u_t = 1 + \int_0^t v_s(x) ds \to e^{-t} \quad (\text{as } r \to \infty). \tag{7}$$

Now we have a conformal flow defined by the sequence of (Σ, q_t, H_t).

In this conformal flow, we can show that

$$\dot{A}_t = 0 \tag{8}$$

and

$$\dot{m}_t \leq 0. \tag{9}$$

Here A_t is the area of H_t and m_t is the ADM mass for (Σ, q_t). When we show Eq. (9), an idea of Bunting and Masood-ul-Alam [4] was used in a crucial way. From these we have

$$A_\infty = A_0 \tag{10}$$

and

$$m_\infty \leq m_0. \tag{11}$$

In the limit of $t = \infty$, we can also show that (Σ, q_t) becomes the Schwarzschild slice. Therefore $\sqrt{A_\infty/16\pi} = m_\infty$ holds. Thus,

$$\sqrt{A_0/16\pi} = \sqrt{A_\infty/16\pi} = m_\infty \le m_0 \tag{12}$$

is proven. This is the Riemannian Penrose inequality.

It is difficult to see why this proof works. So we will modify the proof which is just a reformulation of the conformal flow. Although the new argument requires rather minor technical modifications from Bray's one, we gain a new insight, which in turn offers a physical interpretation to the conformal flow.

III. NORMALIZED CONFORMAL FLOW

Let us introduce the following conformal transformation

$$\tilde{q}_t = \tilde{u}_t^4 q_0, \tag{13}$$

where $\tilde{u}_t$ is defined by

$$\tilde{u}_t = \left(\frac{m_0}{m_t}\right)^{1/2} u_t. \tag{14}$$

u_t is the same with the previous one in Eq. (2). Now we have a new flow $(\Sigma, \tilde{q}_t, H_t)$. Note that the surface H_t remains minimal after the dilation of the metric. It is easy to show

$$\dot{\tilde{m}}_t = 0. \tag{15}$$

In addition,

$$\dot{\tilde{A}}_t = 4\int_{H_t} (\dot{\tilde{u}}_t/\tilde{u}_t) dS. \tag{16}$$

In the integrand,

$$\dot{\tilde{u}}_t = \left(\frac{m_0}{m_t}\right)^{1/2} v_t - \frac{1}{2}\left(\frac{m_0}{m_t}\right)^{1/2} \frac{\dot{m}_t}{m_t} u_t. \tag{17}$$

Since $\dot{m}_t \le 0$

$$\dot{\tilde{u}}_t|_{H_t} = -\frac{1}{2}\left(\frac{m_0}{m_t}\right)^{1/2} \frac{\dot{m}_t}{m_t} u_t|_{H_t} \ge 0. \tag{18}$$

Thus

$$\dot{\tilde{A}}_t \ge 0. \tag{19}$$

We can show that the space becomes the Schwarzschild slice in the $t = \infty$ limit as well as the case of the conformal flow. Thus, $16\pi\tilde{m}_\infty^2 = \tilde{A}_\infty$ holds. Finally we can show the Riemannian Penrose inequality again as

$$16\pi m_0^2 = 16\pi\tilde{m}_\infty^2 = \tilde{A}_\infty \ge A_0. \tag{20}$$

Namely over this normalized conformal flow, the ADM mass is conserved and the area of the apparent horizon is increasing. The former corresponds to the well-known fact that the ADM mass is a conserved quantity in asymptotically flat spacetimes. The latter corresponds to the area theorem of black holes (See Ref. [5] for the area theorem of apparent horizon). These features offers a nice physical interpretation of the normalized conformal flow. In the next section, we will look at this context more closely.

IV. PHYSICAL INTERPRETATION

A. General spacetime

From now on, we will regard the normalised conformal flow as a time evolution. We suppose that the time evolution is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2(t,x)dt^2 + \tilde{q}_t \\ &= -\alpha^2(t,x)dt^2 + \tilde{q}_{tij}dx^i dx^j, \end{aligned} \tag{21}$$

where α is the lapse function and $\tilde{q}_{tij}$ is the component of $\tilde{q}_t$. Later we will choose α so that the $t =$ const. slices are asymptotically flat in the usual sense. In this case the extrinsic curvature of $t =$const. hypersurfaces becomes

$$K_{ij} = \frac{1}{2\alpha}\partial_t \tilde{q}_{tij} = 2\frac{\dot{\tilde{u}}_t}{\alpha\tilde{u}_t}\tilde{q}_{tij}. \tag{22}$$

Then it turns out that the expansion rate θ of the outgoing null geodesic congruence on H_t, (which is by definition, equal to $h^{\mu\nu}\nabla_\mu(t_\nu + r_\nu)$ where r^μ is the unit normal vector to H_t in $(\Sigma, \tilde{q}_t)$, t^μ is the unit coordinate vector, making $t^\mu + r^\mu$ outgoing null vector, $h_{\mu\nu}$ is the induced metric on the surface H_t,) is non-negative

$$\theta|_{H_t} \propto (k + K - K_{ij}r^i r^j)|_{H_t} = -2\frac{\dot{m}_t}{\alpha m_t} \ge 0. \tag{23}$$

This is because of $\dot{m}_t \le 0$(See Eq. (9)). Here k is the trace of extrinsic curvature of H_t with respect to $\tilde{q}_t$ and $K = K^i_i$. Thus H_t is located outside an apparent horizon/marginally trapped surface in a virtual spacetime (M, g).

In the time evolution of H_t, we can see that H_t approaches to the apparent horizon

$$\theta|_{H_t} \propto -2\dot{m}_t/m_t \to 0, \tag{24}$$

because we know that the final state at $t = \infty$ is Schwarzschild slice, the convergence implies $\dot{m}_t \to 0$ as $t \to \infty$.

Let us suppose that (M, g) satisfies the four dimensional Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{25}$$

where $R_{\mu\nu}$ and R are the Ricci curvature and scalar curvature of g. Here we do not yet have the above equation determining the virtual spacetime. The stress tensor $T_{\mu\nu}$ needs to be chosen so that the above equation is satisfied.

To do so, let us focus on the Hamiltonian and momentum constraints,

$$ {}^t\tilde{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho \quad (26)$$

and

$$\tilde{D}^i K_{ij} - \tilde{D}_j K = -8\pi J_j, \quad (27)$$

where $\rho = T_{\mu\nu}t^\mu t^\nu$, $J_i = T_{\mu i}t^\mu$. ${}^t\tilde{R}$ and $\tilde{D}_i$ are the Ricci scalar the covariant derivative with respect to $\tilde{q}_t$, respectively. From the Hamiltonian constraint, we can calculate ρ

$$16\pi\rho = 16\pi\Big(\frac{m_t}{m_0}\Big)^2 u_t^{-4}\rho_0 + 24\frac{1}{\alpha^2}\Big(\frac{\dot{\tilde{u}}_t}{\tilde{u}_t}\Big)^2 \geq 0. \quad (28)$$

In the above we used ${}^0\tilde{R} = 16\pi\rho_0$, where ρ_0 is the energy density of real matters in the physical initial data. Note that ρ_0 is not one computed from virtual matters $T_{\mu\nu}$ here. Then we see that ρ comes out to be non-negative. This is a nice feature in the physical sense.

Next we can calculate J_i and the result is

$$2\pi J_i = \partial_i(v_t/\alpha u_t). \quad (29)$$

On H_t, we have

$$2\pi J_i|_{H_t} = \partial_i v_t/(\alpha u_t)|_{H_t}. \quad (30)$$

Since $v_t(x)$ is the harmonic function, the maximum principle tells us $\partial_i v_t \leq 0$ outward direction of H_t. More precisely, if one introduces the outward normal vector r^i of H_t in t =const. slices, $r^i\partial_i v_t \leq 0$. Thus we can see the ingoing energy flux of artificial matters, that is, $r^iJ_i \leq 0$.

Here note that $K = 6\dot{\tilde{u}}_t/\alpha\tilde{u}_t \simeq 6v_t/\alpha u_t \to -6/\alpha$ as $r \to \infty$. If α is taken to be $\sim r^2$ at $r = \infty$, we can make t =const. slice to be asymptotically flat in the usual way. In the energy density of the virtual matter, the second term of right-hand side of Eq. (28) is proportional to r^{-4}. So it behaves like a radiation. Here we note that T_{ij} is determined by the following algebraic equation for T_{ij}. Note that everything else has been already picked.

$$\begin{aligned}&-\tilde{D}_i\tilde{D}_j\alpha + \alpha({}^t\tilde{R}_{ij} + K^k_k K_{ij} - 2K_{ik}K^k_j) + \dot{K}_{ij}\\ &= 8\pi\alpha\Big(T_{ij} + \frac{1}{2}g_{ij}(\rho - T^k_k)\Big),\end{aligned} \quad (31)$$

where ${}^t\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{q}_t$. Using this, in principle, we can check if the dominant energy condition is satisfied. However, we have to compute the second derivative of u_t, which is included in $\dot{K}_{ij}$, to do so. Unfortunately, the information of the second derivative is not given in the normalised conformal flow. Thus it is difficult to see if the dominant energy condition is satisfied. We would expect that we can choose the lapse function, α, so that the dominant energy condition is satisfied. This issue is beyond of current work.

As a consequence, we have the following physical picture for the normalised conformal flow. The virtual time evolution corresponds to the gravitational collapse. From the behavior of virtual matters characterized by $T_{\mu\nu}$, the 3-dimensional hypersurface $\cup_t H_t$ looks like a horizon. Moreover, the area of H_t is increasing with time. We recall in Bray's construction that the topological type of the surface H_t may change, as the surface may jump across some singular times. And H_t approaches to the horizon because the expansion rate of null congruence on H_t is decaying to zero at $t = \infty$. Thus, the normalized conformal flow gives us a virtual gravitational collapse. Since the final state is promised to be Schwarzschild slice in this evolution, it is natural to have the Penrose inequality. If we know that the final state is Schwarzschild slice, the area theorem implies the Penrose inequality.

B. Example: evolving Schwarzschild slice

As an example, we consider the virtual spacetime modeled by the normalised flow for the Schwarzschild slice.

$$\begin{aligned}ds^2 &= -\alpha^2dt^2 + \tilde{q}_{tij}dx^idx^j\\ &= -\alpha^2dt^2 + \left(e^{-t} + \frac{M}{2r}e^t\right)^4 (dr^2 + r^2d\Omega^2)\,.\end{aligned} \quad (32)$$

Here H_t is located at $r = \frac{M}{2}e^{2t}$. Note that the mass $2(e^{-t})(Me^t/2)$ and the area of $H_t = 16\pi M^2$ are both kept constant in t. Furthermore, we would emphasize that the above spacetime is not obtained by a coordinate change of the Schwarzschild *spacetime* metric and in particular does not satisfy the vacuum Einstein equation. Instead, it will be made to satisfy the non-vacuum Einstein equation *driven by* a suitably chosen Eq. (31) stress-energy tensor $T_{\mu\nu}$ as seen below.

The extrinsic curvature of t = const. hypersurface is

$$K_{ij} = \frac{2}{\alpha}\left(\frac{-e^{-t} + \frac{M}{2r}e^t}{e^{-t} + \frac{M}{2r}e^t}\right) \times \tilde{q}_{tij}. \quad (33)$$

On H_t, we see $K_{ij}|_{H_t} = 0$. This is a peculiar feature for the normalised conformal flow of the Schwarzschild spacetime. Note that our normalization is trivial for $\dot{m} = 0$ in this evolution. This indicates that the horizon does not have a nontrivial time evolution in the current virtual dynamical evolution. Indeed, we can check that that the expansion rate of outgoing null geodesic congruence θ_t vanishes as

$$\theta|_{H_t} \propto \left(k + K - K_{ij}r^ir^j\right)|_{H_t} = 0. \quad (34)$$

Here we used the fact that H_t is the minimal surface and $K_{ij}|_{H_t} = 0$. This means H_t coincide with the apparent horizon of the virtual gravitational collapse throughout the evolution .

From the Hamiltonian constraint of Eq. (26), we can see that the matter density on H_t vanishes as

$$16\pi\rho|_{H_t} = \left({}^t\tilde{R} + K^2 - K_{ij}K^{ij}\right)|_{H_t} = 0. \quad (35)$$

In the above we used ${}^t\tilde{R} = 0$. On the other hand, the 3-momentum of Eq. (27) is evaluated as

$$\begin{aligned} J_r|_{H_t} &= -\frac{1}{8\pi}\left(D^j K_{jr} - D_r K\right)|_{H_t} \\ &= -\frac{e^{-2t}}{2\pi\alpha M} \leq 0. \end{aligned} \tag{36}$$

Thus we see that the artificial matter represented by $T_{\mu\nu}$ has the trivial energy density on H_t. This is merely consistent with the fact that the area of H_t does not increase with the time. On the other hand, it has a nontrivial ingoing (through H_t) 3-momentum J_r.

V. IMPLICATION TO CHARGED BLACK HOLES

Although our new proof is just a rearrangement of Bray's proof, there is a possibility to apply it to other issues. For example, one may want to address the Penrose inequality for charged black holes. According to Ref. [6], Bray's argument is hoped to be generalized so that

$$m_0 \geq m_\infty = \frac{1}{2}\Big(R + \frac{Q^2}{R}\Big) \tag{37}$$

holds where Q is the charge of black holes. The Reissner-Nordström slice realizes the equality. Introducing the area radius by $R = \sqrt{A_0/4\pi} = \sqrt{A_\infty/4\pi}$, the above is rewritten by

$$m_0 - \sqrt{m_0^2 - Q_0^2} \leq R \leq m_0 + \sqrt{m_0^2 - Q_0^2}. \tag{38}$$

However, in Ref. [6], a counterexample to the lower bound was constructed. Because of the evidence, it is unlikely that Bray's proof works for charged black holes in the way presented above.

On the other hand, we may expect that the upper bound for the area radius holds. Namely we hope to show that the inequality

$$4\pi\Big(m_0 + \sqrt{m_0^2 - Q_0^2}\Big)^2 = A_\infty \geq A_0 = 4\pi R^2 \tag{39}$$

(that is $m_0 + \sqrt{m_0^2 - Q_0^2} \geq R$) holds. The lesson to be learned from the counterexample is that in Bray's original flow, the area radius was fixed while the mass was decreased via the flow, though physically the area should be increased till it reaches the maximal value set by the fixed mass. This is what we have done with the normalization. So with charge in play, we may hope to prove with m and Q fixed, the area can be increased till it reaches that of Reissner-Nordström's specified by the parameters (m_0, Q_0).

VI. SUMMARY

In this article, we proposed a proof of the Riemannian Penrose inequality which is a modification of Bray's proof (Ref.[3].) In the original proof by Bray, a conformal flow of the Riemannian metrics was employed, so that the mass is decreasing while the area of the horizon is fixed. However, it is difficult to see the physical reason why the proof works. Hence we proposed a dual viewpoint by normalizing the conformal flow. It is a family of conformal transformations so that now the mass is fixed while the area is increasing. Then we observed that the behaviors of the dual flow enjoy some plausible physical features, that is, the normalised conformal flow corresponds to a virtual time evolution of gravitational collapse, satisfying a non-vacuum Einstein equation. In addition, our new approach may shed some new light to prove the following Penrose type inequality for charged black holes.

$$4\pi\Big(m_0 + \sqrt{m_0^2 - Q_0^2}\Big)^2 \geq A_0, \tag{40}$$

which is consistent with a picture (Ref.[7]) resulting from the cosmic censorship as well as the so-called no-hair theorem where an evolving black hole is expected to settle down to a Kerr(-Newman) spacetime with the parameters (m_0, Q_0) specified by the initial slice. This is left for future study.

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[1] For example, H. L. Bray and P. T. Chrusciel, arXiv:gr-qc/0312047.

[2] G. Huisken and T. Ilmanen, J. Deff. Geo. **59**, 353(2001).

[3] H. L. Bray, J. Diff. Geo. **59**, 177(2001)[arXiv:math/9911173]; arXiv:0902.3241.

[4] G. Bunting and A. Masood-ul-Alam, Gen. Rel. Grav. **19**, 147(1987).

[5] S. A. Hayward, Phys. Rev. D **49**, 6467 (1994).

[6] G. Weinstein and S. Yamada, Comm. Math. Phys. **257**, 703(2005).

[7] R. Penrose, *Unsolved Problems in General Relativity*, Ann. Math. Study **102**, 663(1982).